On the functional equation of transitivity.

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1. The operation $z=x*y$ defined on a set M $[x,y,z\in M]$ will be called *transitive*, if the equation

$$(x*t)*(y*t)=x*y \tag{1}$$

holds. If $x*y=F(x,y)$ is a function of two variables defined on the interval (a,b) of real numbers, with values in (a,b), then (1) takes the form

$$F[F(x,t),F(y,t)]=F(x,y). \tag{1'}$$

The functions $x-y$ and $\frac{x}{y}$ satisfy this equation, which gives thus a direct characterization of the inverse of the group operations without referring to the original operations.[1]) We shall see, that (1) implies the group properties of the inverse operation $x=z\circ y$ of $z=x*y$.[2])

A. R. SCHWEITZER[3]) has solved the functional equation (1′) by reducing it to a differential equation. J. ACZÉL has kindly called my attention to the problem of solving (1′) without supposing differentiability. In the sections 2, 3 of this paper I shall give the same solution of (1′), which A. R. SCHWEITZER has given, but supposing only that $F(x,y)$ is continuous and strictly monotonic. In the section 4 I shall consider the functional equation

$$F[G(x,t),\ H(y,t)]=K(x,y),$$

which is a generalization of (1′), and I shall solve it under suitable hypotheses of differentiability, by reducing it to a differential equation.

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1) (*Added in proof:*) P. LORENZEN [Ein vereinfachtes Axiomensystem für Gruppen, *Journal reine angew. Math.*, **182** (1940), 50] has characterized the inverse of the group operations by (1) and by the solvability of the equation $x_0*y=z$ for at least one x_0.

2) M. WARD [Postulates for the inverse operations in a group, *Transactions of the American Math. Soc.*, **32** (1930), 520—526] gave another elementary characterization by using other postulates instead of (1). He examined also the other inverse operation of $x*y$. It might be remarked that the characteristic properties of this second inverse and the postulates of M. WARD follow also from our investigations.

3) A. R. SCHWEITZER, On a functional equation, *Bulletin of the American Math. Soc.*, **18** (1912), 160—161, 299—302; On the iterative properties of an abstract group, *ibidem*, **24** (1918), 371.

2. First we examine some algebraic properties of the transitive operations.

Theorem I. *Let $z = x * y$ be an operation defined on the set M $[x, y, z \in M]$, suppose it is transitive [cf. (1)], and that the cancellation law holds:*

(2) $$x * y_1 = x * y_2 \quad \textit{for all} \quad x \in M \quad \textit{implies} \quad y_1 = y_2.$$

Then there exists in M a right-hand unit, i. e. an element $e \in M$ such that

(3) $$x * e = x$$

for all $x \in M$, and the operation is an involution, i. e. for any $x \in M$

(4) $$x * x = e$$

holds. Moreover, the operation $z = x * y$ has an inverse operation $x = z \circ y$, and this satisfies the group-axioms.

P r o o f. *a*) First we show (3). Let $x = x_0$ be an arbitrary constant. We define $e = x_0 * x_0$. Putting $x = y = t = x_0$ in (1), we see that e is idempotent:

$$e * e = e.$$

Now let us put $x = t = e$ in (1), then we get

$$e * y = (e * e) * (y * e) = e * (y * e),$$

whence making use of (2), we get

$$y * e = y$$

for any y, and this is (3). (2) involves the unicity of e, since

$$x * e = x = x * e'$$

implies $e = e'$.

b) In order to prove (4), we choose $t = y = x$ in (1); so we get

$$(x * x) * (x * x) = x * x = e_x.$$

Consequently,

$$e_x * e_x = e_x.$$

On the other hand, by (3),

$$e_x * e = e_x,$$

hence by (2) we get $e_x = e$. This proves (4).

We remark that e is no left-hand unit, since putting $t = x$ in (1), we get

$$(x * x) * (y * x) = x * y;$$

i. e., by (4),

(5) $$e * (y * x) = x * y$$

and in general $x * y \neq y * x$.

c) Now we show that the operation

(6) $$z \circ y = z * (e * y)$$

is the inverse of the operation $z=x*y$. Indeed, it follows from (1) and (3) that

(6') $$z\circ y=(x*y)*(e*y)=x*e=x.$$

d) We prove that the operation defined by (6) is a group operation. Since obviously $z\circ y\in M$, we have only to show that

(α) $$x\circ(y\circ t)=(x\circ y)\circ t;$$

(β) there exists a left unit $e\in M$, i. e. $e\circ a=a$ for any $a\in M$;

(γ) there exists for any $a\in M$ a left inverse $x=a^{-1}\in M$, i. e. a solution of the equation $x\circ a=e$.

In order to prove (β) and (γ), we observe that

(δ) the equation $x\circ a=b$ can be solved for arbitrary $a, b\in M$, and the solution is $x=b*a$.

This follows from (6').

(δ) and (4) imply (β), for (δ) gives, putting $b=a$,

$$x=a*a=e,$$

which satisfies (β).

Further, putting $b=e$, (δ) gives

$$x=a^{-1}=e*a,$$

which satisfies the condition (γ).

Finally we verify the equation

$$x*\{e*[y*(e*t)]\}=(x\circ y)*(e*t),$$

which, on account of (6), is equivalent to (α) for arbitrary $x, y, t\in M$.

Denoting here $x\circ y$ by u and $e*t$ by v, and using (δ) we get $x=u*y$, and we have to prove that

$$(u*y)*[e*(y*v)]=u*v.$$

But

(5) $$e*(y*v)=v*y$$

reduces the required equation to (1), which completes the proof of theorem I.

Remark. Theorem I can be inverted as follows:

The inverse operation $z=x*y=x\circ y^{-1}$ of a group operation $x=z\circ y$ defined on a set M is transitive and also (2), (3) *and* (4) *hold.*

(2), (3) and (4) hold evidently. We have only to verify (1) or the equivalent equation

$$(x\circ t^{-1})\circ(y\circ t^{-1})^{-1}=x\circ y^{-1},$$

or, what is the same,

$$x\circ t^{-1}=(x\circ y^{-1})\circ(y\circ t^{-1}).$$

But here, by (α), (γ) and (β), we have

$$(x\circ y^{-1})\circ(y\circ t^{-1})=[(x\circ y^{-1})\circ y]\circ t^{-1}=[x\circ(y^{-1}\circ y)]\circ t^{-1}=(x\circ e)\circ t^{-1}=x\circ t^{-1}$$

thus our statement is proved.

3. L. E. J. Brouwer[4]) has proved the following theorem:

All one-dimensional continuous groups are isomorphic to the addition group of real numbers, i. e. the group operation has the form

$$z \circ y = f^{-1}[f(z)+f(y)] \qquad (\text{"quasi addition"}),$$

where $f(t)$ is an arbitrary continuous, strictly monotonic function with $f(e)=0$ (e denotes the unit element) and $f^{-1}(\tau)$ is its inverse function[5]).

Making use of this theorem and our theorem I, we obtain the following

Theorem II. *In order that the continuous and strictly monotonic function $F(x,y)$ defined on the interval (a,b) of real numbers be transitive, i. e. satisfy the functional equation*

$$F[F(x,t), F(y,t)] = F(x,y) \qquad [x,y,t,F \in (a,b)], \tag{1'}$$

it is necessary and sufficient, that $F(x,y)$ should be written in the form

$$F(x,y) = f^{-1}[f(x)-f(y)] \qquad (\text{"quasi difference"}), \tag{7}$$

where $f(x)$ is an arbitrary continuous and strictly monotonic function with $f(e)=0$.

The conditions of theorem I are fulfilled since, $F(x,y)$ being strictly monotonic,

$$F(x,y_1) = F(x,y_2) \quad \text{for all} \quad x \in M \quad \text{implies} \quad y_1 = y_2. \tag{2'}$$

Remark. A direct proof of theorem II can be obtained by the following recursive construction of $\varphi(t) = f^{-1}(t)$:

a) $\varphi(1) = c$ is arbitrary $(a \leqq c \leqq b)$;

b) if $\varphi\left(\frac{1}{2^{n-1}}\right)$ is already defined for an integer $n \geqq 1$, then $\varphi\left(\frac{1}{2^n}\right)$ is defined by the equation

$$F\left[\varphi\left(\frac{1}{2^{n-1}}\right), \varphi\left(\frac{1}{2^n}\right)\right] = \varphi\left(\frac{1}{2^n}\right);$$

c) if $\varphi\left(\frac{2k-1}{2^n}\right)$ is already defined for two integers n, k with $n \geqq 1$, $k \geqq 1$, then $\varphi\left(\frac{2k+1}{2^n}\right)$ is defined by the equation

$$F\left[\varphi\left(\frac{2k+1}{2^n}\right), \varphi\left(\frac{2k-1}{2^n}\right)\right] = \varphi\left(\frac{1}{2^{n-1}}\right),$$

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[4]) L. E. J. Brouwer, Die Theorie der endlichen kontinuierlichen Gruppen unabhängig von den Axiomen von Lie, *Math. Annalen*, 67 (1909), 246—267.

[5]) In the following we denote the inverse of any strictly monotonic, continuous function $f(t)$ by $f^{-1}(t)$.

and conversely, if $\varphi\left(\frac{2k+1}{2^n}\right)$ is already defined for integer $n \geqq 1$ and $k < 1$, then $\varphi\left(\frac{2k-1}{2^n}\right)$ is defined by the same equation.

Thus $\varphi(t)$ will be defined by induction for all dyadically rational values of t. The functional equation

$$F[\varphi(t), \varphi(\tau)] = \varphi(t-\tau) \tag{7'}$$

is then satisfied for dyadically rational t, τ. We define $\varphi(t)$ for arbitrary values of t as

$$\varphi(t) = \lim_{t_\nu \to t} \varphi(t_\nu)$$

where $\{t_\nu\}$ is a sequence of dyadically rational values tending to t and thus we see that the same equation (7') is also satisfied for arbitrary t, τ.

Finally we get (7) by writing $t = f(x)$ and $\tau = f(y)$.

We omit the further details of this direct proof.[6])

4. We turn to the generalization of (1').

Theorem III. *All strictly monotonic and continuously differentiable solutions of the functional equation*

$$F[G(x,t), H(y,t)] = K(x,y) \tag{8}$$

can be written in the form

$$\begin{cases} F(x,y) = h[\varphi(x) - \psi(y)], \\ K(x,y) = h[f(x) - g(y)], \\ G(x,y) = \varphi^{-1}[f(x) - k(y)], \\ H(x,y) = \psi^{-1}[g(x) - k(y)] \end{cases} \tag{9}$$

where $h(t)$, $\varphi(t)$, $\psi(t)$, $f(t)$, $g(t)$, $k(t)$ are arbitrary strictly monotonic functions with continuous derivatives.

Proof. Differentiating (8) with respect to the variable t we get

$$F_1[G(x,t), H(y,t)]\, G_2(x,t) + F_2[G(x,t), H(y,t)]\, H_2(y,t) = 0$$

where the indices 1 and 2 denote the partial differential quotient with respect to the first and second variable, respectively.

Now, choosing arbitrarily a value t_0 of t, we define the functions $\varphi(u)$ and $\psi(u)$ by the equations

$$\varphi'[G(x;t_0)] = G_2(x,t_0), \quad \psi'[H(y,t_0)] = H_2(y,t_0),$$

then by writing x for $G(x,t_0)$ and y for $H(y,t_0)$ we get

$$F_1(x,y)\,\varphi'(x) + F_2(x,y)\,\psi'(y) = 0,$$

[6]) We do not treat, for exemple, the existence of $\lim\limits_{t_\nu \to t} \varphi(t_\nu)$ and the solvability of the equations $F(x,y) = z_0$ and $F(x_0, y) = y$, which were used at the recursive definition of $\varphi(t)$ in *b*), *c*).

i. e. the Jacobian of $F(x, y)$ and $\varphi(x)-\psi(y)$ is equal to 0. Thus the functions $F(x, y)$ and $\varphi(x)-\psi(y)$ are dependent:

$$F(x, y)=h[\varphi(x)-\psi(y)].$$

Substituting this into (8) we get

$$h\{\varphi[G(x, t)]-\psi[H(y, t)]\}=K(x, y), \tag{10}$$

and putting

$$f(x)=\varphi[G(x, t_0)], \quad g(y)=\psi[H(y, t_0)]$$

we arrive to the relation

$$K(x, y)=h[f(x)-g(y)].$$

Further we use the notations

$$r(x)=h^{-1}[K(x, y_0)], \quad s(t)=\psi[H(y_0, t)],$$
$$\varrho(y)=-h^{-1}[K(x_0, y)], \quad \sigma(t)=\varphi[G(x_0, t)].$$

For $y=y_0$ resp. $x=x_0$ (10) gives

$$\varphi[G(x, t)]=r(x)+s(t),$$
$$\psi[H(y, t)]=\varrho(y)+\sigma(t),$$

respectively.

If we put our results into (8), we obtain

$$h[r(x)+s(t)-\varrho(y)-\sigma(t)]=h[f(x)-g(y)],$$

hence the solution satisfies (8) only if

$$r(x)=f(x)+c_1, \; \varrho(y)=g(y)+c_2, \; \sigma(t)=s(t)+c_3$$

hold with $c_1=c_2+c_3$. Denoting here

$$-k(t)=s(t)+c_1=\sigma(t)+c_2$$

we see that

$$G(x, y)=\varphi^{-1}[f(x)-k(y)], \quad H(x, y)=\psi^{-1}[g(x)-k(y)],$$

and this completes the solution (9).

To complete the proof of the theorem III we have only to observe that the functions (9) satisfy the equation (8).

(Received December 6, 1953.)